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Schottky uniformization and the symplectic structure of the cotangent bundle of a Teichmüller space

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Abstract

Using the Schottky uniformization of a marked Riemann surface, the space of equivalence classes of complex projective structures on a compact oriented C^∞ surface gets identified, in an obvious fashion, with the total space of the holomorphic cotangent bundle of the corresponding Teichmüller space. This identification is proved to be compatible with the natural symplectic structures on these two spaces. If we consider the other identification of the same two spaces obtained using the Bers' construction of universal Riemann surface, then the compatibility of the symplectic structures was established by Kawai [S. Kawai, Math. Ann. 305 (1996) 161–182]. © 2000 Elsevier Science B.V. All rights reserved.

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1. Introduction

Let S be a compact oriented C^∞ surface of genus g, with $g \geq 2$. Let $\mathrm{Diff}_0(S)$ denote the group of diffeomorphisms of S homotopic to the identity map of S. For the natural action of $\mathrm{Diff}_0(S)$ on the space of conformal structures, $\mathrm{Conf}(S)$, on S, compatible with the orientation, the quotient $\mathrm{Conf}(S)/\mathrm{Diff}_0(S)$ is the Teichmüller space of S, which will be denoted by $\mathcal{T}(S)$.

A *projective structure* on S is defined by giving a covering of S by coordinate charts of the form (U, ϕ), where ϕ is an orientation preserving smooth map from an open subset U of S to an open subset of $\mathbb{C}$, such that every transition function is a restriction of some Möbius transformation [3]. Recall that a Möbius transformation is a function of the form $z \mapsto (az+b)/(cz+d)$, where $a, b, c, d \in \mathbb{C}$ and $ad - bc = 1$.

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Let Möb(S) denote the space of all projective structures on S. The quotient space Möb$(S)/\mathrm{Diff}_0(S)$, which is a complex manifold of dimension $6g-6$, will be denoted by $\mathcal{P}(S)$.

The natural projection

$$\psi : \mathcal{P}(S) \to \mathcal{T}(S) \tag{1.1}$$

is a torsor for the holomorphic cotangent bundle $T^*\mathcal{T}(S)$. This means that for any point $[X] \in \mathcal{T}(S)$, representing a Riemann surface X, the cotangent space $T^*_{[X]}\mathcal{T}(S)$, which is $H^0(X, K_X^{\otimes 2})$, acts freely transitively on the fiber $\psi^{-1}([X])$ [3]. More precisely, the fiber is an affine space for the vector space $H^0(X, K_X^{\otimes 2})$.

Given a C^∞ section

$$f : \mathcal{T}(S) \to \mathcal{P}(S) \tag{1.2}$$

of ψ, i.e., $\psi \circ f$ is the identity map of $\mathcal{T}(S)$, we have a C^∞ diffeomorphism

$$L_f : T^*\mathcal{T}(S) \to \mathcal{P}(S) \tag{1.3}$$

which send any cotangent vector $\omega \in T_t^*\mathcal{T}(S)$, over $t \in \mathcal{T}(S)$, to the projective structure $f(t) + \omega$ on the Riemann surface represented by the point t.

Using the developing map for a projective structure, we have a map

$$D : \mathcal{P}(S) \to \mathcal{R} := \frac{\mathrm{Hom}(\pi_1(S), \mathrm{PSL}(2, \mathbb{C}))}{\mathrm{PSL}(2, \mathbb{C})},$$

which is locally a biholomorphism [5,6]. The image of the map D is contained in the smooth locus of $\mathbb{R}$. Let $\Omega_{\mathcal{P}}$ denote the holomorphic symplectic structure on $\mathcal{P}(S)$ obtained by pulling back, using the map D, the natural symplectic structure on the smooth locus of $\mathcal{R}$. Given a representation $\rho \in \mathcal{R}$, let $\mathrm{Ad}(\rho)$ denote the flat vector bundle on S associated to ρ for the adjoint representation of $\mathrm{PSL}(2, \mathbb{C})$ on its Lie algebra. If $\underline{\mathrm{Ad}(\rho)}$ denotes the associated local system, then

$$T_\rho\mathcal{R} = H^1(S, \underline{\mathrm{Ad}(\rho)}),$$

and the pairing

$$\alpha, \beta \mapsto \int_S \mathrm{trace}(\alpha \cup \beta),$$

where $\alpha, \beta \in H^1(S, \underline{\mathrm{Ad}(\rho)})$, defines a symplectic structure on the smooth locus of $\mathcal{R}$ which was constructed in [1,2].

Let Ω denote the natural symplectic structure on the cotangent bundle $T^*\mathcal{T}(S)$.

Using his idea of simultaneous uniformization, Bers constructed a universal Riemann surface over $\mathcal{T}(S)$, which gave a section of the map ψ defined in (1.1). Let h denote this section of ψ. The main theorem of [7, p. 165], says that the equality

$$\pi L_h^*\Omega_{\mathcal{P}} = \Omega \tag{1.4}$$

is valid, where L_h is constructed in (1.3).

A *Schottky group* $G \subset \mathrm{PSL}(2, \mathbb{C})$ is a finitely generated, free, purely loxodromic Kleinian group [8]. For any compact Riemann surface X of genus g, there is a Schottky group G, freely generated by g elements, say $\{\gamma_1, \gamma_2, \dots, \gamma_g\}$, such that quotient $\Omega(G)/G$ is isomorphic to X, where $\Omega(G) \subset \mathbb{CP}^1$ is the region of discontinuity for G. The quotient space $\Omega(G)/G$ has a natural projective structure, with the inclusion map of $\Omega(G)$ in $\mathbb{CP}^1$ as the developing map.

Choose a marking on X, i.e., fix $a_1, \dots, a_g, b_1, \dots, b_g \in \pi_1(X, x_0)$ such that $\pi_1(X, x_0)$ is the free group generated by $\{a_1, \dots, a_g, b_1, \dots, b_g\}$ quotiented by its normal subgroup generated by $\prod_{i=1}^{g} a_i b_i a_i^{-1} b_i^{-1}$. Then there is a Schottky group G uniformizing X, as sketched above, such that the image of G in $\pi_1(X, x_0)$ is the normal subgroup generated by $\{a_1, \dots, a_g\}$. Furthermore, such a group G is unique up to an inner conjugation by an element of $\mathrm{PSL}(2, \mathbb{C})$. Since an inner conjugation by an element of $\mathrm{PSL}(2, \mathbb{C})$ does not alter the projective structure, the projective structure on X is uniquely determined.

Therefore, using the above construction of a projective structure from the Schottky uniformization, we have a section s of the submersion ψ (constructed in (1.1)) as in (1.2). This section s is known to be holomorphic. Our aim here is to prove the following analog of the theorem of Kawai [7].

Theorem 1.5. *Let s denote the section of ψ obtained from the Schottky uniformization. Then the equality*

$$\pi L_s^* \Omega_{\mathcal{P}} = \Omega$$

is valid.

The above theorem will be proved in Section 2 after establishing a criterion for the validity of the equality (1.4) for a section f of ψ. The criterion is expressed in terms of the difference between f and the C^∞ section of ψ given by the Fuchsian uniformization (Lemma 2.7).

2. A criterion for compatibility of the symplectic structures

Continuing with the notation of Section 1, our first step would be to establish the following lemma.

Lemma 2.1. *Given a C^∞ section f of the submersion ψ, the equality*

$$\pi L_f^* \Omega_{\mathcal{P}} = \Omega$$

is valid if and only if the 2-form $f^* \Omega_{\mathcal{P}}$ on $\mathcal{T}(S)$ vanishes identically.

Proof. Take a C^∞ (1,0)-form θ on $\mathcal{T}(S)$. Let

$$t(\theta) : T^* \mathcal{T}(S) \to T^* \mathcal{T}(S)$$

be the translation map, which sends a cotangent vector $\omega \in T_z^*\mathcal{T}(S)$, at a point $z \in \mathcal{T}(S)$, to the cotangent vector $\omega + \theta(z)$.

We first want to show that

$$t(\theta)^*\Omega = \Omega + p^*\mathrm{d}\theta, \tag{2.2}$$

where $p : T^*\mathcal{T}(S) \to \mathcal{T}(S)$ is the natural projection.

The above assertion is evident. Indeed, $\Omega = \mathrm{d}\alpha$, where α denotes the tautological 1-form on $T^*\mathcal{T}(S)$. It is immediate from the definition of α that $t(\theta)^*\alpha = \alpha + p^*\theta$. This implies the equality in (2.2).

Recall the section h of ψ, constructed by Bers, that was used in (1.4). Now, for the given section f, set θ to be the $(1, 0)$-form defined by the identity

$$h(z) + \theta(z) = f(z)$$

valid for all $z \in \mathcal{T}(S)$.

Therefore, we have the following commutative diagram of maps:

$$\begin{array}{ccc} T^*\mathcal{T}(S) & \xrightarrow{L_h} & \mathcal{P}(S) \\ \downarrow t(\theta) & & \downarrow \\ T^*\mathcal{T}(S) & \xrightarrow{L_f} & \mathcal{P}(S) \end{array} \tag{2.3}$$

where the above diffeomorphism of $\mathcal{P}(S)$ is defined by sending $y \in \psi^{-1}(z)$ for $z \in \mathcal{T}(S)$, to the projective structure $y + \theta(z)$.

Therefore, in view of the equality (2.2), in order to complete the proof of the lemma it suffices to show that the equality in the statement of the lemma is valid if and only if $\mathrm{d}\theta = 0$.

Consider the map $\overline{\theta} : \mathcal{T}(S) \to T^*\mathcal{T}(S)$ defined by θ, which sends any $z \in \mathcal{T}(S)$ to the cotangent vector $\theta(z) \in T_z^*\mathcal{T}(S)$. Since the pullback of Ω to $\mathcal{T}(S)$, by using the zero section, vanishes identically, the equality (2.2) implies that

$$\mathrm{d}\theta = \overline{\theta}^*\Omega. \tag{2.4}$$

Now from the result of Kawai, stated in (1.4), and the commutativity of the diagram (2.3), it follows immediately that the equality

$$\overline{\theta}^*\Omega = \pi f^*\Omega_{\mathcal{P}} \tag{2.5}$$

is valid. Combining (2.4) and (2.5) we conclude that $\mathrm{d}\theta = 0$ if and only if $f^*\Omega_{\mathcal{P}} = 0$. This completes the proof of the lemma. □

Since from the uniformization theorem we know that any Riemann surface X, of genus at least two, is a quotient of the upper half plane by a discrete torsion-free subgroup of $\mathrm{PSL}(2, \mathbb{R})$, which is unique up to an inner conjugation, X has a natural projective structure. Consequently, we get a C^∞ section τ, of the submersion ψ in (1.1), obtained from the Fuchsian uniformization.

For a C^∞ section f of ψ, let λ_f denote the smooth $(1,0)$-form on $\mathcal{T}(S)$ defined by the condition

$$\tau(z) + \lambda_f(z) = f(z) \quad \text{for all } z \in \mathcal{T}(S), \tag{2.6}$$

where τ is the section given by the Fuchsian uniformization.

Lemma 2.7. *Let f be a holomorphic section of the submersion* ψ in (1.1). *The equality*

$$\pi L_f^* \Omega_{\mathcal{P}} = \Omega$$

is valid if and only if the $(2,0)$*-form* $\partial\lambda_f$ *on* $\mathcal{T}(S)$ *vanishes identically.*

Proof. Let θ_τ denote the smooth $(1,0)$-form on $\mathcal{T}(S)$ defined by the condition $\tau = \theta_\tau + h$, where h, as before, is the section of ψ constructed by Bers.

Combining (2.4) and (2.5) we have $\mathrm{d}\theta/\pi = f^*\Omega_{\mathcal{P}}$. Consequently, we have the following equality:

$$f^*\Omega_{\mathcal{P}} - \tau^*\Omega_{\mathcal{P}} = \frac{\mathrm{d}\theta - \mathrm{d}\theta_\tau}{\pi} = \frac{\mathrm{d}\lambda_f}{\pi}, \tag{2.8}$$

where θ_τ is defined above.

Since $\Omega_{\mathcal{P}}$ is a holomorphic symplectic form, and the section f is assumed to be holomorphic, the form $f^*\Omega_{\mathcal{P}}$ is holomorphic of type $(2,0)$.

In view of Lemma 2.1 and the equality obtained by taking the $(2,0)$ component of the two sides of the equality (2.8), to complete the proof of the lemma it is enough to show that

$$(\tau^*\Omega_{\mathcal{P}})^{2,0} = 0, \tag{2.9}$$

where $(\tau^*\Omega_{\mathcal{P}})^{2,0}$ is the $(2,0)$ type component of the 2-form $\tau^*\Omega_{\mathcal{P}}$. Indeed, if (2.9) is valid, then the $(2,0)$ type component of the equality (2.8) would become $f^*\Omega_{\mathcal{P}} = \partial\lambda_f/\pi$.

Firstly, (2.4) and (2.5) together imply that $\mathrm{d}\theta_\tau/\pi = \tau^*\Omega_{\mathcal{P}}$. Hence $\tau^*\Omega_{\mathcal{P}}$ is a sum of a $(2,0)$-form, namely $\partial\theta_\tau/\pi$, and a $(1,1)$-form, namely $\bar\partial\theta_\tau/\pi$. Secondly, since the image of τ is contained in the smooth locus of the subset

$$\frac{\mathrm{Hom}(\pi_1(S), \mathrm{PSL}(2,\mathbb{R}))}{\mathrm{PSL}(2,\mathbb{R})} \subset \frac{\mathrm{Hom}(\pi_1(S), \mathrm{PSL}(2,\mathbb{C}))}{\mathrm{PSL}(2,\mathbb{C})},$$

and the pullback of Ω to the smooth locus of $\mathrm{Hom}(\pi_1(S), \mathrm{PSL}(2,\mathbb{R}))/\mathrm{PSL}(2,\mathbb{R})$ by the inclusion map is a real 2-form, the identity $\overline{\tau^*\Omega_{\mathcal{P}}} = \tau^*\Omega_{\mathcal{P}}$ is valid. From these two observations we conclude that $\tau^*\Omega_{\mathcal{P}}$ must be of type $(1,1)$. In fact, in [2, Section 2] it has been proved that $\tau^*\Omega_{\mathcal{P}}$ is a constant scalar multiple of the Weil-Petersson Kähler form on $\mathcal{T}(S)$. This establishes the equality (2.9), and the proof of the lemma is complete. □

Now we are in a position to prove Theorem 1.5.

Proof of Theorem 1.5. We will show that the section s defined by the Schottky uniformization satisfies the criterion in Lemma 2.7. Firstly, the section s is holomorphic [4,5,9].

Now, the main result of [9, Remark 2, p. 310] says that

$$\lambda_s = \frac{1}{2}\partial S,$$

where λ_s is the $(1, 0)$-form constructed in (2.6) for the section of ψ defined by the Schottky uniformization, and S is a C^∞ function on $\mathcal{T}(S)$ constructed in [9], which is the action function for Liouville equation.

Therefore, $\partial\lambda_s = 0$, and the proof of Theorem1.5 is complete. □

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